CHAPTER 4

HIGH ORDER DIFFERENTIAL EQUATION

1. Introduction to Linear Higher Order Equations

$$y_G = y_C + y_P$$

$y_C \rightarrow$ Complementary solution

$y_P \rightarrow$ Particular solution

1. The roots are real and different

$$y_C = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots$$

2. The roots are real and repeat

$$y_C = e^{\lambda x}[C_1 + C_2 x + C_3 x^2 + \cdots]$$

3. The roots are not real and different

$$\alpha \pm i\beta$$

$$y_C = e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x + \cdots]$$

4. The roots are not real and repeat

$$y_C = e^{\alpha x}\{[C_1 + C_2 x + \cdots] \cos \beta x + [C_3 + C_4 x + \cdots] \sin \beta x\}$$

EXAMPLE1►

Find the solution of differential equation

$$\boldsymbol{y'' + 3y' - 4y = 0}$$

SOLUTION

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 0$$

$$y\left[\frac{d^2}{dx^2}+3\frac{d}{dx}-4\right]y=0,$$

$$[D^2+3D-4]y=0, \qquad D=\frac{d}{dx}$$

Auxiliary equation

$$\lambda^2+3\lambda-4=0$$
$$(\lambda+4)(\lambda-1)=0$$
$$\lambda=-4, \qquad \lambda=1$$

The complementary solution is

$$y_C=C_1e^{-4x}+C_2e^{x}$$

EXAMPLE 2►

Find the solution of differential equation

$$\boldsymbol{y'''-8y''+16y=0}$$

SOLUTION

$$[D^4-8D+16]y=0, \qquad D=\frac{d}{dx}$$

Auxiliary equation

$$\lambda^4-8\lambda+16=0$$
$$(\lambda^2-4)(\lambda^2-4)=0$$
$$\lambda=\pm2, \qquad \lambda=\pm2$$

The complementary solution is

$$y_C=e^{2x}[C_1+C_2x]+e^{-2x}[C_3+C_4x]$$

EXAMPLE 3►

Find the solution of differential equation

$$\boldsymbol{y''+2y'+5y=0}$$

SOLUTION

$$[D^2 + 2D + 5]y = 0, \qquad D = \frac{d}{dx}$$

Auxiliary equation

$$\lambda^2 + 2\lambda + 5 = 0$$

$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 \times 5}}{2}$$

$$\lambda = -1 \pm 2i$$

The complementary solution is

$$y_C = e^{-x}[C_1 \cos 2x + C_2 \sin 2x]$$

EXAMPLE 4►

Find the solution of differential equation

$$\boldsymbol{y^{(4)} + 8y^{(2)} + 16y = 0}$$

SOLUTION

$$[D^4 + 8D + 16]y = 0, \qquad D = \frac{d}{dx}$$

Auxiliary equation

$$\lambda^4 + 8\lambda + 16 = 0$$

$$(\lambda^2 + 4)(\lambda^2 + 4) = 0$$

$$\lambda = \pm 2i, \qquad \lambda = \pm 2i$$

The complementary solution is

$$y_C = [C_1 + C_2 x] \cos 2x + [\, C_3 + C_4 x] \sin 2x\}$$

2. Undetermined Coefficients for Higher Order Equations

$$\boldsymbol{ay'' + by' + cy = f(x)}$$

If $f(x) = ax + b$,	$y_P = px + q$
If $f(x) = ax^2 + bx + c$,	$y_P = px^2 + qx + r$
If $f(x) = e^{ax}$,	$y_P = pe^{ax}$
If $f(x) = \sin ax$,	$y_P = p \sin ax + q \cos ax$

EXAMPLE 1►

Find the general solution of the differential equation

$$\boldsymbol{y'' - y = x^2}$$

SOLUTION

$$[D^2 - 1]y = 0, \qquad D = \frac{d}{dx}$$

Auxiliary equation

$$\lambda^2 - 1 = 0$$
$$(\lambda - 1)(\lambda + 1) = 0$$
$$\lambda = 1, \qquad \lambda = -1$$

The complementary solution is

$$y_C = C_1 e^x + C_2 e^{-x}$$

The particular solution is

$$y_P = px^2 + qx + r$$
$$y_P' = 2px + q$$
$$y_P'' = 2p$$

Substitute into this equation
$\boldsymbol{y'' - y = x^2}$ then

$$2p - px^2 - qx - r = x^2$$

By comparing the coefficients

$$-2p - r = 0, \qquad p = -1, \ r = -2, \quad q = 0.$$

$$y_p = -x^2 - 2$$

The general solution is

$$y_G = y_C + y_P$$
$$y_G = C_1 e^x + C_2 e^{-x} - x^2 - 2$$

EXAMPLE 2►

Find the general solution of the differential equation

$$\boldsymbol{y''' - y'' + 4y' - 4y = \cos x}$$

SOLUTION

$$[D^3 - D^2 + 4D - 4]y = 0, \qquad D = \frac{d}{dx}$$

Auxiliary equation

$$\lambda^3 - \lambda^2 + 4y - 4 = 0$$
$$y^2(\lambda - 1) + 4(\lambda - 1) = 0$$
$$(\lambda - 1)(y^2 + 1) = 0$$
$$\lambda = 1, \qquad \lambda = \pm 2i$$

The complementary solution is

$$y_C = C_1 e^x + C_2 \cos 2x + C_3 \sin 2x$$

The particular solution is

$$y_P = p \cos x + q \sin x$$
$$y_P' = -p \sin x + q \cos x$$
$$y_P'' = -p \cos x - q \sin x$$
$$y_P''' = p \sin x - q \cos x$$

Substitute into this equation
$\boldsymbol{y''' - y''+4y' - 4y = \cos x}$ then
$p \sin x - q \cos x + p \cos x + q \sin x - 4p \sin x + 4q \cos x - 4p \cos x - 4q \sin x = \cos x$

By comparing the coefficients

$$-3p - 3q = 0, \qquad 3q - 3p-= 1, \;\; p = \frac{-1}{6}, \quad q = \frac{1}{6}.$$

$$y_p = \frac{-1}{6}\cos x + \frac{1}{6}\sin x$$

The general solution is

$$y_G = y_C + y_P$$

$$y_G = C_1 e^x + C_2 \cos 2x + C_3 \sin 2x \frac{-1}{6}\cos x + \frac{1}{6}\sin x$$

EXAMPLE 3►

Find the general solution of the differential equation

$$\boldsymbol{y''+2y' - 3y = 10e^{2x}}$$

SOLUTION